Microeconomics II: Exercise from Problem Set 2

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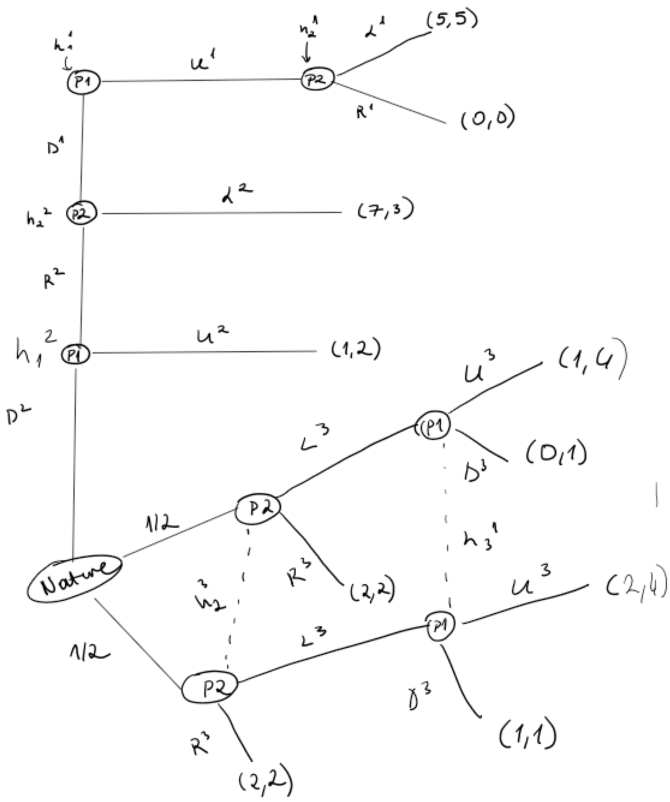
Problem Set 2: Problems on Dynamic Games

Exercise 1. Extensive Form

In the game below find: the normal form, all pure and mixed NE and all SPNE.

The extensive form representation is below. h_i^k represents the k^{th} information set of player k.



1) *The normal form.*

Constructing the normal form:

(a) Any strategy profile that has $(U^1 \cdot\cdot, R^1 \cdot\cdot)$[1] will have (0,0): Player 1 plays U in the first move, P2 plays R1, game is over. Fill these in first.

(b) Any strategy profile that has $(U^1 \cdot\cdot, L^1 \cdot\cdot)$ will have (5,5).

(c) Otherwise, we are in strategy profiles where P1 played D1. If Player 2 passes, game over: anything with $(D^1 \cdot\cdot, \cdot L^2 \cdot)$ has (7,3)

(d) Otherwise, we are in $(D^1 \cdot\cdot, \cdot R^2 \cdot)$ and the game continues. U^2 ends the game with (1,2), all profiles with $(D^1 U^2 \cdot, \cdot R^2 \cdot)$ is filled with (1,2).

(e) If P1 plays D^2, Nature gets to randomize now. Player 2 playing R^3 ends the game with (2,2):$(D^1 D^2 \cdot, \cdot R^2 R^3)$ is $(2,2)$

(f) We have two empty cells left for the case where P2 plays L^3, and P1 gets to play. Now we need to take expectations since Nature is randomizing: U^3 yields $1/2(2,4) + 1/2(1,4) = (1.5,4)$, D^3 yields $1/2(0,1) + 1/2(1,1) = (0.5,1)$.

The normal form of the game as a whole:

		P2							
		$R^1R^2R^3$	$R^1R^2L^3$	$R^1L^2R^3$	$R^1L^2L^3$	$L^1R^2R^3$	$L^1R^2L^3$	$L^1L^2R^3$	$L^1L^2L^3$
	$U^1U^2U^3$	0 , 0	0 , 0	0 , 0	0 , 0	$\underline{5}$, $\underline{5}$	$\underline{5}$, $\underline{5}$	5 , $\underline{5}$	5 , $\underline{5}$
	$U^1U^2D^3$	0 , 0	0 , 0	0 , 0	0 , 0	$\underline{5}$, $\underline{5}$	$\underline{5}$, $\underline{5}$	5 , $\underline{5}$	5 , $\underline{5}$
	$U^1D^2U^3$	0 , 0	0 , 0	0 , 0	0 , 0	$\underline{5}$, $\underline{5}$	$\underline{5}$, $\underline{5}$	5 , $\underline{5}$	5 , $\underline{5}$
	$U^1D^2D^3$	0 , 0	0 , 0	0 , 0	0 , 0	$\underline{5}$, $\underline{5}$	$\underline{5}$, $\underline{5}$	5 , $\underline{5}$	5 , $\underline{5}$
P1	$D^1U^2U^3$	1 , 2	1 , 2	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$	1 , 2	1 , 2	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$
	$D^1U^2D^3$	1 , 2	1 , 2	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$	1 , 2	1 , 2	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$
	$D^1D^2U^3$	$\underline{2}$, 2	$\underline{1.5}$, $\underline{4}$	$\underline{7}$, 3	$\underline{7}$, 3	2 , 2	1.5 , $\underline{4}$	$\underline{7}$, 3	$\underline{7}$, 3
	$D^1D^2D^3$	$\underline{2}$, 2	0.5 , 1	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$	2 , 2	0.5 , 1	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$

2) *All pure and mixed Nash equilibria.*

Pure strategies:
The normal form of the game contains the BR of each player underlined. There is a considerable number of equilibria, so let us use a more concise formulation.

A strategy s_i contains an action for each player i in each information set: $s_i = (a_i^1, a_i^2, a_i^3)$:

$$\begin{aligned} PSNE = \{(s_1^*, s_2^*)\} = &\left\{\left(U^1 \cdot\cdot\right), \left(L^1 R^2 \cdot\right)\right\} \\ &\cup \left\{\left(\left(D^1 U^2 \cdot\right) \cup \left(D^1 D^2 D^3\right)\right), \left(\cdot L^2 \cdot\right)\right\} \\ &\cup \left\{\left(D^1 D^2 U^3\right), \left(R^1 R^2 L^3\right)\right\} \end{aligned}$$

Mixed strategies: To simplify the search of mixed strategies, we can write a condensed normal form of the game. Note that I omit strategy $R^1R^2R^3$ for player two as it is strictly dominated by $L^1L^2L^3$. Again, the BRs of each player are underlined. (Note how they coincide with those of the original game).

[1] $\cdot$ refers to player i playing either of his two available actions.

		P2				
		$R^1R^2L^3$	$R^1L^2\cdot$	$L^1R^2R^3$	$L^1R^2L^3$	$L^1L^2\cdot$
	$U^1\cdot\cdot$	0 , 0	0 , 0	$\underline{5}$, $\underline{5}$	$\underline{5}$, $\underline{5}$	5 , $\underline{5}$
	$D^1U^2\cdot$	1 , 2	$\underline{7}$, $\underline{3}$	1 , 2	1 , 2	$\underline{7}$, $\underline{3}$
P1	$D^1D^2U^3$	$\underline{1.5}$, $\underline{4}$	$\underline{7}$, 3	2 , 2	1.5 , $\underline{4}$	$\underline{7}$, 3
	$D^1D^2D^3$	0.5 , 1	$\underline{7}$, $\underline{3}$	2 , 2	0.5 , 1	$\underline{7}$, $\underline{3}$

Before we start to derive indifference conditions for player 1 observe that strategies $D^1U^2\cdot$ and $D^1D^2D^3$ are weakly dominated by $D^1D^2U^3$. Hence, we can't have strategies in which P2 totally mixes. We can see that by setting the indifference condition between each pair of dominated and dominant strategy. Before we do that, denote probabilities of P2 playing each strategy as: $Pr(R^1R^2L^3) = \alpha_1$, $Pr(R^1L^2\cdot) = \alpha_2, \ldots,$ $Pr(L^1L^2\cdot) = \alpha_5 = 1 - \sum_{i=1}^{4} \alpha_i$.
Let's start with setting up an indifference condition between $D^1U^2\cdot$ and $D^1D^2U^3$:

$$1.5\alpha_1 + 7\alpha_2 + 2\alpha_3 + 1.5\alpha_4 + 7\alpha_5 = \alpha_1 + 7\alpha_2 + \alpha_3 + \alpha_4 + 7\alpha_5$$
$$2\alpha_3 = -\alpha_4 - \alpha_1 \implies \alpha_3 = \alpha_4 = \alpha_1 = 0$$

If we set up an indifference condition between $D^1D^2U^3$ and $D^1D^2D^3$ we will arrive with $\alpha_1 + \alpha_4 = 0$ which implies $\alpha_4 = \alpha_1 = 0$. This makes perfect sense as for these cases P1 is only indifferent when P2 plays $R^1L^2\cdot$ or $L^1R^2L^3$. However, would P2 be willing to do that? Observe that she gets a profitable deviation by mixing $R^1R^2L^3$ and $L^1R^2L^3$. We have to check then for which α_1, α_4 P1 would be indifferent or prefer to play $D^1D^2U^3$ than $U^1\cdot\cdot$:

$$1.5\alpha_1 + 1.5\alpha_4 \geq 5\alpha_4$$
$$\alpha_1 + \alpha_4 = 1$$
$$\implies \alpha_1 \geq \frac{7}{10},\ \alpha_4 \leq \frac{3}{10}$$

Hence, we got our first set of MSNE.
Also, $U^{\cdot}\cdot$ is not weakly dominated by any other strategy of P1. However, player 1 would be willing to play it with probability 1 if and only if $\alpha_1 = \alpha_2 = 0$ and:

$$5\alpha_3 + 5\alpha_4 + 5(1 - \alpha_3 - \alpha_4) \geq 2\alpha_3 + 1.5\alpha_4 + 7(1 - \alpha_3 - \alpha_4)$$
$$10\alpha_3 + 11\alpha_4 \geq 4$$

Any values which satisfy the above condition and $\alpha_3 + \alpha_4 + \alpha_5 = 1$ will constitute a MSNE, for instance: $\alpha_3 = \frac{2}{5}, \alpha_4 = \frac{1}{11}, \alpha_5 = \frac{56}{110}$. If you need to provide just a sample of MSNE, you can assume that α_3 or α_4 is equal to zero and solve the equation with only one variable (I showed this method during the class). But this does not exhaust the whole set of possible MSNE.

Let's now consider player 2 and denote probabilities of playing each strategy by P1 as $Pr(U^1\cdot\cdot) = \beta_1, \ldots,$ $Pr(D^1D^2D^3) = \beta_4 = 1 - \sum_{i=1}^{3} \beta_i$. Strategies $R^1L^2\cdot$ and $L^1R^2R^3$ are weakly dominated by $L^1L^2\cdot$ and if we set up an indifference condition between each of weakly dominant and dominated strategy we will arrive at contradiction: either $\beta_1 = 1$ or $\beta_2 + \beta_3 + \beta_4 = 1$.
However, notice that we can make P2 play $R^1L^2\cdot$ with probability equal to 1 if P1 mixes strategies

$D^1 \cdot\cdot$ in such a way that:

$$3\beta_2 + 3\beta_3 + 3(1-\beta_2-\beta_3) \geq 2\beta_2 + 4\beta_3 + (1-\beta_2-\beta_3)$$
$$\beta_2 + 3\beta_3 \leq 2$$

Hence, for every $\beta_2, \beta_3, \beta_4 = 1-\beta_2-\beta_3$ satisfying the above condition we can find a MSNE. One example is: $\beta_2 = \beta_3 = \frac{1}{2}$. Notice that nobody would have then a profitable deviation.
Note that if $\beta_1 = 0$ the above solution is equivalent to the situation in which P1 would like player 2 to play $L^1L^2\cdot$ with probability 1 (and we would compare payoffs from playing $L^1L^2\cdot$ and $L^1R^2L^3$).

We can characterize the set of MSNE in the following way:

$$\begin{aligned} MSNE = \Big\{ &\left(\sigma_{U^1..} = 1, \sigma_{R^1.} = 0, 10\sigma_{L^1R^2R^3} + 11\sigma_{L^1R^3L^3} \geq 4, \sigma_{L^1L^2.} = 1 - \sigma_{L^1R^2R^3} - \sigma_{L^1R^3L^3}\right), \\ &\left(\sigma_{D^1D^2U^3} = 1, \sigma_{R^1L^2.} = \sigma_{L^1R^2R^3} = \sigma_{L^1L^2.} = 0, \sigma_{R^1R^3L^3} \geq \frac{7}{10}, \sigma_{L^1R^2L^3} = 1 - \sigma_{R^1R^2L^3}\right), \\ &\left(\sigma_{U^1..} = 0, \sigma_{D^1U^2.} + 3\sigma_{D^1D^2U^3} \leq 2, \sigma_{D^1U^2.} + \sigma_{D^1D^2U^3} + \sigma_{D^1D^2D^3} = 1, \sigma_{L^1L^2.} = 1\right), \\ &\left(\sigma_{U^1..} = 0, \sigma_{D^1U^2.} + 3\sigma_{D^1D^2U^3} \leq 2, \sigma_{D^1U^2.} + \sigma_{D^1D^2U^3} + \sigma_{D^1D^2D^3} = 1, \sigma_{R^1L^2.} = 1\right), PSNE \end{aligned}$$