Static Games: Example

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Voting by parties

In this example, there are two parties, one which forms a government (G) and one opposition (O). They vote on a policy that is largely supported both by voters and the government (e.g., tax cuts on food), but the opposition is known to be against it. The ruling party forms the majority in parliament. When both parties vote on the policy, the payoff for the government is 100 (the policy they support is implemented), but for the opposition, it is zero (they would lose anyway, but at least they gained popularity among voters). If the opposition votes in favor of the policy but the government is against the policy, the payoff for the opposition is 50 (it voted against its' convictions, but the policy is not implemented anyway because the government voted against it and they gained the support from voters). When the opposition votes against the policy and the government is in favor, the payoff for the opposition is zero (they voted in line with their convictions but lost) and 100 for the government. When both vote against, the payoff for the opposition is 100 (they won and voted in line with their convictions), and for the government, it is -100.

We can construct the following normal/strategic representation of the game:

Table 0.1: Normal Form of the Game

Opposition \ Government	For	Against
For	0,100	50, -100
Against	0,100	100,-100

Weak/strict dominance

We can easily notice that the government has a strictly dominant voting strategy in favor of the policy. For the opposition, the weakly dominant strategy is to vote against it. If we employ iterated elimination of weakly dominated strategies and start with eliminating weakly dominated strategies of opposition, we would end up with a Nash equilibrium (A,F). But this is not the only Nash equilibrium: if we start the iteration of strictly dominated strategies with the government, we will end up with two equilibria: (F,F) and (A,F). We ended up with two strategies, so the game is not dominance solvable. However, both of them are Nash equilibria, which we show below.

Nash equilibria

Let's first find all PSNE by looking at the BR of each player to the other player's strategies.

If the government plays "For", the opposition is indifferent between for and against, and when the government plays against, the opposition would also vote against. On the other hand, when the opposition votes for or against the government, it always plays for. We have the following candidates for the PSNE: {(F,F),(A,F)}.

- Is (F, F) a Nash equilibrium? Yes.

    1. There are no profitable unilateral deviations for either player from (F, F),

because:

$$u_G(F,F) \geq u_G(A,F)$$
$$u_O(F,F) \geq u_O(A,F)$$

2. $BR_G(F) = \{F\}, BR_O(F) = \Delta\{A,F\}$.

- Is (A,F) a Nash equilibrium? Yes.

1. There are no profitable unilateral deviations for either player from (A,F), because:

$$u_G(F,A) \geq u_G(F,A)$$
$$u_O(A,F) \geq u_O(F,F)$$

2. $BR_G(A) = F, BR_O(F) = \Delta\{A,F\}$.

- (F,A) cannot be a Nash Equilibrium, because:

1. $u_G(F,A) > u_G(A,A)$: the government has a profitable deviation.
2. $A \notin BR_G(A)$, the government's best response to A is not A, but F.

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Now, let's consider mixed strategies. Before considering indifference conditions, we can conclude that there will be no MSNE as the government strictly prefers to vote in favor of the policy. Also, for the opposition, the strategy to vote against is weakly dominant, so the government must only use degenerate probabilities. We can also show it with calculations. Denote the probability of the government choosing strategy F as p and the opposition choosing strategy F as q. The indifference condition for the opposition is then the following:

$$\begin{aligned} 50(1-p) &= 100(1-p) \\ &\implies p = 1 \end{aligned}$$

And for the government:

$$\begin{aligned} 100q + 100(1-q) &= -100q - 100(1-q) \\ &\implies 200 \neq 0 \end{aligned}$$

There is no q that would make the government indifferent between voting in favor or against the policy. Also, only for $p = 1$, the opposition is indifferent between their two strategies. Hence, there is no MSNE.

Trembling hand

We know that every finite normal game has a THP Nash equilibrium. We also know that a Nash equilibrium containing weakly dominated strategies cannot be THP. From this, we can deduce that (F,F) is not THP, and that (A,F) is. Let's show it by using sequences.

Add a slight tremble to the N.E. (F,F) and denote the sequence of totally mixed strategy profiles as $\sigma_1^k(F), \sigma_2^k(F) = (1 - \epsilon_k^1, 1 - \epsilon_k^2)$. What is the intuition behind the sequence of strategies? Think of it as a one-shot game, but played k times, and each time by different players (so we need $2k$ players in total). Each game is independent of the other. As the game is played many times, there might be a non-zero probability of committing a mistake by either of the players. Trembling hand perfect equilibrium is a refinement of N.E. in which the other player will not change her best response even if somebody makes a mistake. Hence, $\sigma^k \to \sigma$. We will see that for the N.E. (F,F) it is not the case.

Then, we can write down the utility of opposition from playing F and A as:

$$u_O(F) = 50\epsilon_2^k \ \forall k$$
$$u_O(A) = 100\epsilon_2^k \ \forall k$$
$$\implies u_O(A) > u_O(F), \ \ \epsilon_2^k > 0 \ \forall k.$$

Hence, as soon as there is a non-zero probability that the government will tremble and play against the opposition, it will switch its strategy and play against it, too. No sequence of totally mixed strategies would converge to the Nash equilibrium (F,F). Hence, (F,F) is not THP Nash equilibrium. Let's now verify that (A,F) is a THP Nash equilibrium. Denote the sequence of totally mixed strategy profiles as $\sigma_1^k(A), \sigma_2^k(F) = (1 - \epsilon_k^1, 1 - \epsilon_k^2)$. Then, we can write down the utility of opposition from playing F and A as:

$$u_O(F) = 50\epsilon_2^k \ \forall k$$
$$u_O(A) = 100\epsilon_2^k \ \forall k$$
$$\implies u_O(A) > u_O(F) \text{ if } \epsilon_2^k > 0 \ \forall k$$

For all values of tremble larger than zero, the opposition would not switch its' strategy. Let's now consider the government and write down its' utilities from playing both strategies:

$$u_G(F) = 100(1 - \epsilon_1^k) + 100\epsilon_1^k$$
$$u_G(A) = -100(1 - \epsilon_1^k) - 100\epsilon_1^k$$
$$u_G(F) > u_G(A) \ \epsilon_k^1 \in [0, 1] \ \forall k$$

For all values of a tremble, the government would not switch its' strategy and will continue to play F. We can now construct a sequence of totally mixed strategies which converges to the Nash equilibrium (A,F): $\sigma_1^k(A), \sigma_2^k(F) = (1 - (\frac{1}{3})^k, 1 - (\frac{2}{3})^k)$. This is just an arbitrary sequence. You can construct any on your own.

Question 1: How would you change the payoffs so that for some values of ϵ_k the government or opposition would change their strategy in the case of the THP equilibrium?

Consider the normal game with the following payoffs (abstracting from discussion whether they make "political sense"):

Table 0.2: Normal Form of the Game

Government / Opposition	For	Against
For	0,100	110, -100
Against	10,100	100,-100

Note that there is now only one PSNE (A,F). We will now show that not for all trembles of government, the opposition will remain voting against (but it will still be a THP!):

$$u_O(F) = 110\epsilon_2$$

$$u_O(A) = 10(1-\epsilon_2) + 100\epsilon_2$$

$$u_O(A) > u_O(F) \iff$$

$$10(1-\epsilon_2) + 100\epsilon_2 > 110\epsilon_2$$

$$\epsilon_2 < \frac{1}{2}$$

Hence, only for trembles smaller than 0.5, the opposition would vote against it. Note that for player 2, nothing changes, and he will vote "for" no matter the value of a tremble. We can construct the following sequence converging to the Nash equilibrium: $\sigma_1^k(A), \sigma_2^k(F) = (1-(\frac{1}{3})^k, 1-(\frac{1}{10})^k)$.

Question 2: What is the difference between THP Nash equilibrium and epsilon-Nash equilibrium?

In a nutshell, epsilon-Nash equilibrium can give payoffs that are within ϵ distance to the optimal payoffs. One reason why optimal payoffs might not be reached is the “status-quo” bias. An article that explains this concept well is “Limit games and limit equilibria“ by Drew Fudenberg and David Levine (Journal of Economic Theory, 1986). Roy Radner uses a good example of this concept in the paper “Collusive behavior in noncooperative epsilon-equilibria of oligopolies with long but finite lives” (Journal of Economic Theory, 1980).

Correlated equilibrium

Now imagine that the parties are voting on a proposition that would increase the salaries of party members. On the one hand, the increase in salaries would increase the welfare of parliamentarians, but on the other, they risk losing support from voters.

The payoffs from each strategy are the following: if both parties vote in favor, each gets 50. If the opposition party votes in favor, but the government is against, the former gets the payoff -10 (not only did it lose in voting, but also lost some support from voters), and the latter +100 (it won the voting and got a lot of support from voters). If the opposition votes against but the ruling party are in favor, the opposition gets +25 (it doesn't win but gains some support from voters), and the government gets -10. If both are against, they get 20 (both win the voting and gain some support from voters). The normal/strategic representation of this game is the following:

Table 0.3:

Government / Opposition	For	Against
For	50,50	-10,100
Against	25,-10	20,20

Note that the unique Nash equilibrium of this game is (A,A). However, playing F by both parties constitutes a Pareto improvement.

Suppose that members of both parties receive cues from their leaders one minute before the actual voting. If the leader of the ruling party is in favor of the policy, she takes off her glasses, and when against it, she rubs her back. If the leader of the opposition party is in favor of the policy, he puts his phone upside down, but when he is against it, he checks his tie. The probability distribution of the leaders' cues is the following:

Table 0.4:

Opposition \ Government	Taking off glasses (F)	Rubbing back (A)
Phone upside down (F)	1/2	1/4
Checking tie (A)	1/4	0

Let's see whether it is a correlated equilibrium. Firstly, we need to the set of outcomes and associated probability measures:

1. $\Omega = \{FF, FA, AF, AA\}$
2. $\pi(FF) = \frac{1}{2},\ \pi(FA) = \frac{1}{4},\ \pi(AF) = \frac{1}{4},\ \pi(AA) = 0$

Secondly, we need to define information partitions:

1. $\mathcal{P}_{\mathcal{G}} = \{(FF, FA), (AF, AA)\}$
2. $\mathcal{P}_{\mathcal{O}} = \{(FF, FA), (AF, AA)\}$

Note that the government does not know the strategies used by the opposition and vice versa. What are the BR of the government and opposition after receiving the cues? Let's start with the opposition.

Opposition

1. The opposition uses Bayesian updating to infer the probability of being in each state after receiving a cue from their leader:

$$\begin{aligned} Pr(FF|F) = \frac{Pr(F|FF)Pr(FF)}{Pr(F)} &= \frac{1 * 1/2}{3/4} = \frac{2}{3} \\ Pr(FA|F) &= \frac{1 * 1/4}{3/4} = \frac{1}{3} \end{aligned}$$

   And the expected payoffs associated with each strategy:

$$\begin{aligned} u_O(F) &= 50 * \frac{2}{3} - 10 * \frac{1}{3} = 30 \\ u_O(A) &= 25 * \frac{2}{3} + 20 * \frac{1}{3} = 23\frac{1}{3} \\ u_O(F) > u_O(A) &\implies BR_O(F) = F \end{aligned}$$

   Hence, the best response of the opposition when their leader puts their phone upside down is to vote "For".

2. Now we'll evaluate what is the best response of the opposition when their leader checks his tie.

$$\begin{aligned} Pr(AF|A) &= \frac{1 * 1/4}{1/4} = 1 \\ Pr(AA|A) &= 0 \end{aligned}$$

   And the expected payoffs associated with each strategy:

$$\begin{aligned} u_O(A) &= 25 \\ u_O(F) &= 50 \\ u_O(F) > u_O(A) &\implies BR_O(A) = F \end{aligned}$$

   Hence, the above **does not** constitute a correlated equilibrium. Even if the opposition's leader checks his tie, the party members vote in favor of the policy.

Question 3: How would you check the probability of the leaders' cues so that we could construct a correlated equilibrium?

In my opinion, it is not possible because playing "Against" by the government is a strictly dominant strategy. Hence, no matter the received cues, it will always play against the policy.

Quantal response equilibrium

[1] The intuition behind the quantal response equilibrium (QRE) concept is similar as in discrete choice models. In both contexts, we assume that individuals have observed and unobserved components in their utility function. The difference in the QRE is that not only an econometrician does not observe this component, but also other players.

Here we have two players (government and opposition): $I = \{1, 2\}$, each having to choose a strategy out of two possibilities: $J = \{1, 2\}$.

Σ_i is the set of probability distributions over actions A_i and $\sigma_i \in \Sigma_i$ is a mixed strategy. For a strategy profile $\sigma \in \Sigma = \Pi_{i \in I}\Sigma_i$ a player i expected payoff would be $v_i(\sigma) = \sum_{a \in A} Pr(a)u_i(a)$ where $Pr(a) = \Pi_{i \in I}\sigma_i(a_i)$.

For each i and each $j \in \{1, 2\}$ and for any $\sigma \in \Sigma$, denote by $v_{ij}(\sigma)$ the expected utility of player i of adopting the pure strategy a_{ij} when the other players use σ_{-i}. For a quantal response equilibrium, it is assumed that for each pure strategy a_{ij}, player i receives an additional privately observed disturbance to their payoff ϵ_{ij}. Then, the payoff of a player of adopting strategy a_{ij} under strategy profile σ is:

$$\hat{v}_{ij} = v(\sigma)_{ij} + \epsilon_{ij}$$

where for each player i's profile of disturbances $\epsilon_i = (\epsilon_{i1}, \epsilon_{i2})$ has joint distribution $f_i(\epsilon_i)$. Assumption: marginal distribution f_i exists, and the expected value of ϵ_i

[1] This section is based on "Chapter 60 Quantal Response Equilibria: A Brief Synopsis" by Richard D. McKelvey, Thomas R. Palfrey, Handbook of Experimental Economics Results, 2008

is zero.

For any v we define $B_{ij}(v)$ as the set of realization of ϵ_i such that strategy a_{ij} has the highest "disturbed" expected payoff. So $P_{ij}(v) = \int_{B_{ij}(v)} f(\epsilon)d\epsilon$ is the induced probability that a player i will select strategy j given v.

Any fixed point σ^* s.t. $\sigma^* = P(v(\sigma^*))$ is a quantal response equilibrium of the game (I, A, u).

Consider once again the same game:

Table 0.5:

Government / Opposition	For	Against
For	50,50	-10,100
Against	15,-10	20,20

As in the lecture, let's consider *logistic* quantal response function.

Propensity to play strategy $\sigma_i j$:

$$p_i(\sigma_{ij}) = exp(\lambda u_i(\sigma_{ij}, \sigma_{-i}))$$

Probability that a strategy σ_{ij} would be played:

$$\pi_i(\sigma_{ij}) = \frac{p_i(\sigma_{ij})}{p_i(\sigma_{ij}) + p_i(\sigma_{i-j})}$$

Now, if $\lambda \to \infty$, $\pi(.)$ approaches the best response; and if $\lambda \to 0$, $\pi(.)$ approaches uniform distribution.

Firstly, consider the opposition:

$$p_O(For) = exp(\lambda(50\sigma_G(For) - 10(1 - \sigma_G(For)))) = exp(\lambda(-10 + 60\sigma_G(For)))$$
$$p_O(Against) = exp(\lambda(15\sigma_G(For) + 20(1 - \sigma_G(For) = exp(\lambda(20 - 5\sigma_G(For)))$$
$$\pi_O(For) = \frac{exp(\lambda(-10 + 60\sigma_G(For)))}{exp(\lambda(-10 + 60\sigma_G(For))) + exp(\lambda(20 - 5\sigma_G(For)))} =$$
$$= \frac{1}{1 + exp(\lambda(30 - 65\sigma_G(For)))}$$
$$\pi_O(Against) = \frac{exp(\lambda(20 - 5\sigma_G(For)))}{exp(\lambda(-10 + 60\sigma_G(For))) + exp(\lambda(20 - 5\sigma_G(For)))} =$$
$$= \frac{1}{1 + exp(\lambda(-30 + 65\sigma_G(For)))}$$

Let's now consider the government:

$$p_G(For) = exp(\lambda(50\sigma_O(For) - 10(1 - \sigma_O(For)))) = exp(\lambda(-10 + 60\sigma_O(For)))$$
$$p_G(Against) = exp(\lambda(100\sigma_O(For) + 20(1 - \sigma_O(For)))) = exp(\lambda(20 + 80\sigma_O(For)))$$
$$\pi_G(For) = \frac{exp(\lambda(-10 + 60\sigma_O(For)))}{exp(\lambda(-10 + 60\sigma_O(For))) + exp(\lambda(20 + 80\sigma_O(For)))} =$$
$$\frac{1}{1 + exp(\lambda(30 + 20\sigma_O(For)))}$$
$$\pi_G(Against) = \frac{exp(\lambda(20 + 80\sigma_O(For)))}{exp(\lambda(-10 + 60\sigma_O(For))) + exp(\lambda(20 + 80\sigma_O(For)))} =$$
$$\frac{1}{1 + exp(\lambda(-30 - 20\sigma_O(For)))}$$

We already deduced that $\sigma_G(For) = 0$, so we can't make the government indifferent between playing For and Against. Then, there is no MSNE, even though we can make the opposition indifferent if the government plays $\sigma_G(For) = \frac{6}{13}$.

If we sketch the BR of both players and quantal probabilities for some values of λ, we can see that quantal responses approach BR for higher values of λ, as expected:

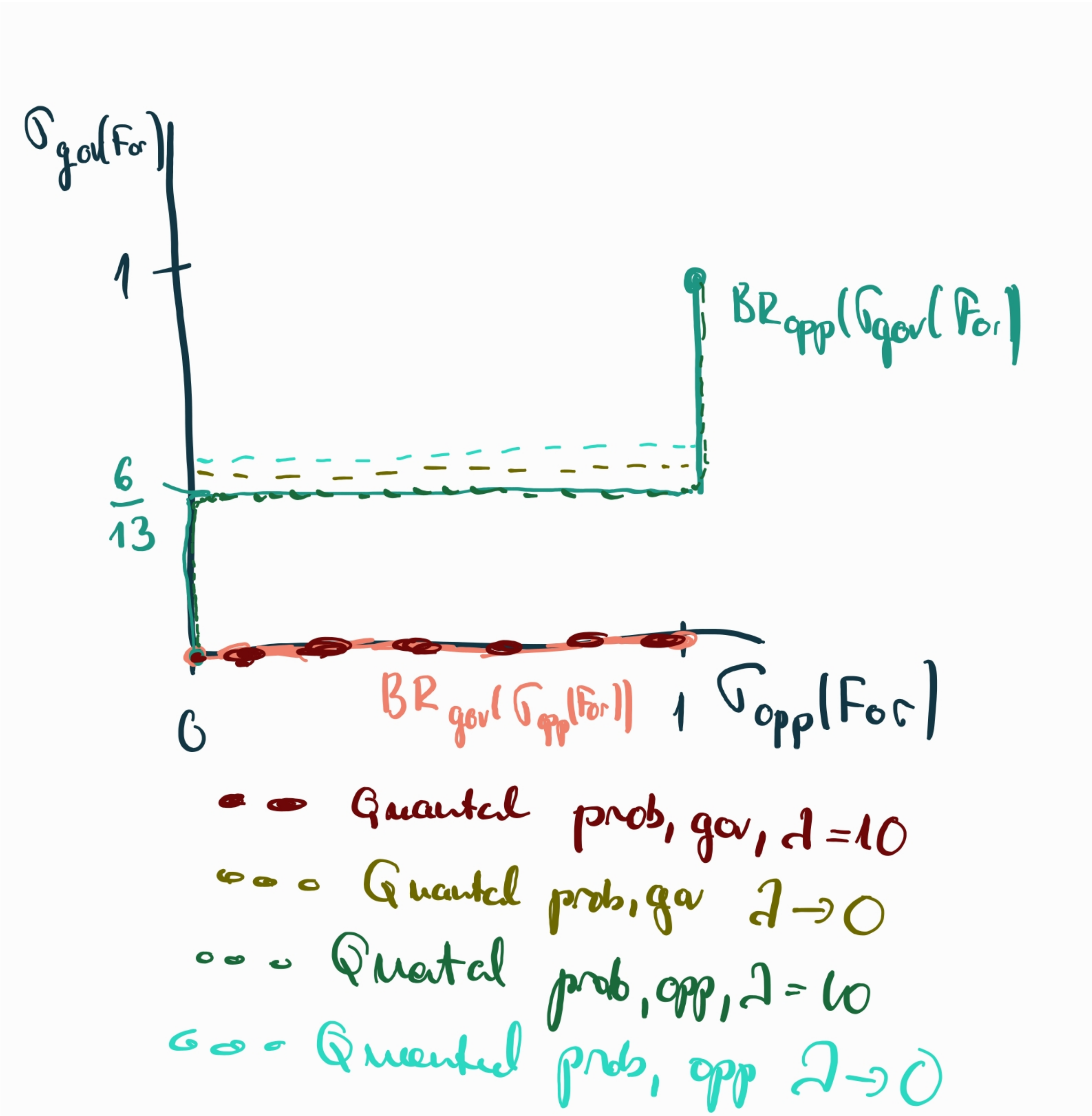

Figure 0.1: Sketch of BR of opposition and government and quantal probabilities